# INFORMATION STRUCTURE AND EQUILIBRIUM ASSET PRICES†

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# Information Structure and Asset Prices

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ABSTRACT

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In a continuous trading economy, it is shown that if information is revealed continuously and if agents' preferences are continuous in a certain topology, then equilibrium asset price processes must have continuous sample paths. Except for uninteresting cases, the sample paths of price processes will be of unbounded variation. In particular, if the information is generated by a Brownian motion, then equilibrium asset price processes are Ito integrals. When information is not revealed continuously, the times (which may be random) at which prices can have jumps are identified. J. Econ. Theory, (English). Massachusetts Institute of Technology, Cambridge, Massachusetts, USA.

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## 1. Introduction and Summary

Among the central topics in the study of financial economics over the past decade have been the characterization of an individual's optimal intertemporal consumption and investment decision and the implications for asset prices in an equilibrium setting. Continuous time stochastic models have been prevalent in that body of literature. Merton [29] studied the consumption-investment problem of an investor in a market where asset prices are Ito integrals. Various equilibrium formulations of financial markets using continuous time models are usually termed Intertemporal Capital Asset Pricing Models. Merton [30] has the seminal paper in this direction. Breeden [4] extended Merton's [30] results.

A common practice in these Intertemporal Asset Pricing Models is to assume that equilibrium asset prices are Ito integrals.[1] Since the purpose of these asset pricing models is merely to characterize properties of equilibrium asset prices, assuming that asset prices are Ito integrals to start with is inappropriate for their purpose. In order to convert these asset pricing models into true general equilibrium models, conditions on the exogenous uncertain environment and on agents' preferences must be found ensuring that equilibrium asset prices are indeed Ito integrals.

The main purpose of this paper is to develop just such a set of primitive assumptions guaranteeing that equilibrium asset prices are Ito integrals. More generally, this paper provides foundational work linking various patterns of information flow to different behaviors of equilibrium asset prices. If agents' preferences are "continuous" and if information is revealed in a "continuous" fashion (the usage of "continuous" in these two contexts to be made precise), then a basic result of this paper is that equilibrium asset price processes will have continuous sample paths. Furthermore, the sample paths are of unbounded variation except for uninteresting cases. In particular, it is shown that an

information structure generated by a Brownian motion is continuous, and for this information structure, asset prices are Ito integrals. For other discontinuous information structures, the times (which may be random) at which asset price processes can have jumps are identified.

In Section 2, a model of a continuous-time frictionless pure exchange economy under uncertainty is formulated. Taken as a primitive is a probability space $(\Omega,\mathcal{F},P)$. Each generic element $\omega\varepsilon\Omega$ denotes a complete description of the exogenous environment. The set of trading dates is the interval $[0,T]$. It is assumed that there is a single perishable consumption commodity consumed only at times 0 and T. The commodity space is taken to be $V \equiv R x L^{\infty}(\Omega,\mathcal{F},P)$, where $L^{\infty}(\Omega,\mathcal{F},P)$ is the space of essentially bounded random variables defined on $(\Omega,\mathcal{F},P)$. Each generic element $(r,x)\varepsilon R x L^{\infty}(\Omega,\mathcal{F},P)$ means r units of consumption at time zero and $x(\omega)$ units of consumption at time T in state $\omega$. There are a finite number of agents in the economy indexed by $i=1,2,\ldots,I$. Each agent is characterized by a preference relation $\gtrsim_i$ on V, the space of <u>net trade</u>. We assume that $\gtrsim_i$ is continuous in a sense to be formalized.

The common information structure of agents, $\mathbb{F} = \{\mathcal{F}_t, t\varepsilon[0,T]\}$, which plays an essential role in this paper, specifies distinguishable events at each $t\varepsilon[0,T]$. Roughly, for each $t\varepsilon[0,T]$, $\mathcal{F}_t$ is the collection of events whose occurrence and non-occurrence can be determined at time t. Putting it another way, $\mathbb{F}$ specifies how events are revealed to agents over time. If, for example, the information structure is generated by some stochastic process then, loosely speaking, $\mathbb{F}$ at t describes all the possible realizations of the stochastic process in the time interval $[0,t]$.

It is assumed that there are $(N+1)$ traded claims in zero net supply, indexed by $n=0,1,\ldots,N$. Each one is characterized by its payoff structure at time T. Each agent's problem is to find a strategy of buying and selling those traded

claims in order to maximize his or her preferences on net trade. The equilibrium concept used is the Equilibrium of Plans, Prices, and Price Expectations of Radner [35].

All the results in this paper are valid if agents are endowed with different probability measures on $(\Omega,\mathcal{F})$, provided each agent's endowed probability measure is equivalent to P. Two probability measures on the measurable space $(\Omega,\mathcal{F})$ are said to be equivalent if they have the same null sets. For expository purpose, however, we shall assume that agents are endowed with a common probability measure P.

The main purpose of this paper is to characterize properties of equilibrium asset prices when an equilibrium indeed exists. In Sections 3 and 4, we shall assume that an equilibrium exists and denote the equilibrium price system by S. Thus $S(\omega,t)$ is the $(N+1)$-dimensional vector of equilibrium prices for the $N+1$ traded claims at time t in state $\omega$.

Section 3 extends Harrison and Kreps [20] in the following direction. In that paper, agents are allowed to trade at only a finite number of prespecified times. The net trade space is $R\times L^2(\Omega,\mathcal{F},P)$, where $L^2(\Omega,\mathcal{F},P)$ is the space of square-integrable random variables on $(\Omega,\mathcal{F},P)$. If there exists a claim, the numeraire security, whose price lies in a compact subinterval of $(0,\infty)$ on $[0,T]$, then by choosing the price of this claim as the numeraire, they showed that the normalized equilibrium price system is a vector of martingales under a substitution of an equivalent probability measure. By allowing only essentially bounded claims, this section validates the martingale result of Harrison and Kreps [20] in our economy where trading can take place continuously. That is, if we denote the vector of normalized equilibrium prices by $S^*$, then a probability measure Q on $(\Omega,\mathcal{F})$ equivalent to P can be constructed such that $E^*(S^*(s)|\mathcal{F}_t) = S^*(t)$ Q-a.s. for $T \geq s \geq t \geq 0$, where $E^*(\cdot)$ is the expectation under Q (Theorem 3.1). We shall call this property the martingale representation property of

equilibrium prices.

A definition for a continuous information structure is given in Section 4, one originally due to Harrison [19]. An information structure $\mathbb{F}$ is continuous if agents endowed with $\mathbb{F}$ update their posterior probability assessments for any event $B \varepsilon \mathcal{F}$ in a continuous fashion. That is, for every distinguishable event $B \varepsilon \mathcal{F}$, the mapping $t \to P(B|\mathcal{F}_t)$ is, with P-probability one, a continuous function of time, where $P(B|\mathcal{F}_t)$ denotes the posterior probability for event B at time t. This definition has two very important consequences. The first is that any martingale adapted to $\mathbb{F}$ is indistinguishable from a continuous process (Proposition 4.1). (The words adapted and indistinguishable will be defined in the sequel.)

Now let us note that the definition for a continuous information structure is given with respect to the reference probability measure P. Intuition suggests that whether or not an information structure is continuous should depend largely upon how events are revealed but not upon the particular probability measure involved. A second important consequence is therefore: an information structure $\mathbb{F}$ is continuous under the probability measure P if and only if it is continuous under any probability measure Q equivalent to P (Proposition 4.2).

Armed with the martingale representation property of equilibrium prices, the continuity of the sample paths of equilibrium asset prices, denominated in units of the numeraire security, is derived directly from the two above-mentioned consequences of a continuous information structure. The argument goes as follows: Suppose that $\mathbb{F}$ is continuous under P. Then it is continuous under Q by Proposition 4.2. By Proposition 4.1 we know that any martingale under Q is indistinguishable from a continuous process. From the martingale representation property of equilibrium asset prices, $S^*$ is a vector martingale under Q. Therefore $S^*$ is indistinguishable from a vector of continuous processes. Since, in an economic equilibrium, only relative prices are determined, and the normalized price system $S^*$ is a vector of relative prices, we are done. Continuous information implies

that $S^*$ must have continuous sample paths.

Furthermore, the continuity of the information structure also requires that the sample paths of equilibrium prices have unbounded variation except for uninteresting cases (Theorem 4.2). This follows from a well-known result in probability theory: a continuous martingale is either of unbounded variation or is a constant (Fisk [15]). Conversely, when the securities markets are complete, in a sense to be defined, then the continuity of prices also implies the continuity of the information structure (Theorem 4.3). Intuitively, we need "enough" securities to distinguish each piece of information coming in.

All of the above results are of course provisional upon the existence of an equilibrium. A general existence proof is beyond the scope of this paper. An autarchy example of the economy with which we are dealing is given in Section 5. The existence of an equilibrium is first established. It is shown next that an information structure generated by a Brownian motion is continuous, and in that case, equilibrium asset prices are Ito integrals. This follows from a celebrated result in probability theory: any martingale adapted to the information structure generated by a Brownian motion can be represented by an Ito integral (cf. Clark [7]).

Section 6 discusses other information structures and links these structures to different behaviors of equilibrium prices. This section ends by showing that an information structure is continuous if and only if no events are "surprises" in a sense to be made precise. Section 7 contains some concluding remarks.

## 2. The Formulation

This section presents a model of a continuous-time frictionless pure exchange economy under uncertainty. Let $(\Omega,\mathcal{F},P)$ be a complete probability space. Each $\omega\varepsilon\Omega$ denotes a complete description of the exogenous environment. The set of trading dates is $\underline{T} = [0,T]$, where T is a strictly positive real number.

Agents in the economy are endowed with a common information structure, which is specified exogenously. Formally the information structure is a family of increasing sub-Borel-fields of $\mathcal{F}$: $\mathbb{F} = \{\mathcal{F}_t, t\varepsilon\underline{T}\}$ with $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$. It is assumed that $\mathcal{F}_0$ is almost trivial[2], that $\mathcal{F}_T = \mathcal{F}$, and that the filtration $\mathbb{F}$ is right-continuous, that is, $\mathcal{F}_t = \bigcap_{s>t}\mathcal{F}_s$ for every $t\varepsilon[0,T)$. Also, $\mathcal{F}_0$ is assumed to contain all P-null events. A process $Y = \{Y(t), t\varepsilon\underline{T}\}$ is said to be adapted to $\mathbb{F}$, or simply adapted, if as a mapping $Y: \Omega \times [0,T] \to R$, Y is measurable with respect to $\mathcal{F}\times B([0,T])$,[3] and if Y(t) is measurable with respect to $\mathcal{F}_t$ for every $t\varepsilon\underline{T}$. The common probability measure on $(\Omega,\mathcal{F})$ held by the agents is denoted by P. The expectation operator under P is denoted by $E(\cdot)$.

There is one perishable consumption commodity consumed only at times zero and T. The net trade space is taken to be $V \equiv R\times X$, where $X = L^\infty(\Omega,\mathcal{F},P)$ is the space of essentially bounded random variables. Thus $(r,x)\varepsilon V$ denotes r units of consumption at time zero and $x(\omega)$ units of consumption at time T in state $\omega$. We endow V with a product topology $\tau$, generated by the Euclidean topology on R and the $L^1$-Mackey topology on X. The $L^1$-Mackey topology is the strongest topology on X such that its topological dual is $L^1(\Omega,\mathcal{F},P)$.[4]

There are $N+1$ consumption claims indexed by $n=0,1,\ldots,N$. Each claim n is represented by a random variable $d_n\varepsilon X$. Interpret $d_n(\omega)$ the number of units of the consumption commodity to which the bearer of one share of claim n is entitled at time T in state $\omega$. Let $X_+ = \{x\varepsilon X: P\{x \geq 0\} = 1\}$, the positive orthant of X, and $X_{++} = \{x\varepsilon X: P\{x \geq c\} = 1 \text{ for some } c\varepsilon R_+, c \neq 0\}$. We assume that $d_0\varepsilon X_{++}$, that is, the payoff of the $0^{th}$ claim is bounded below away from zero. The claims are of zero net supply.

There are a finite number of agents in the economy indexed by $i=1,2,\dots,I$. Each agent i is characterized by a preference relation $\gtrsim_i$ on V. The preference relation $\gtrsim_i$ is assumed to satisfy the following three conditions. First, it is convex, that is, for all $v \varepsilon V$, the set

$$\{v' \varepsilon V : v' \gtrsim_i v\} \tag{2.1}$$

is convex. Second, it is continuous in the following sense. For all $v \varepsilon V$, the sets

$$\begin{gathered}\{v' \varepsilon V : v' \gtrsim_i v\} \\ \text{and} \\ \{v' \varepsilon V : v \gtrsim_i v'\}\end{gathered} \tag{2.2}$$

are $\tau$-closed. Third, it is strictly increasing in the following sense. Let $v' \varepsilon V_+$ and $v' \neq 0$, where $V_+ \equiv R_+ \times X_+$, the positive orthant of V. Then for all $v \varepsilon V$,

$$v + v' >_i v \ ,$$

where, in the usual fashion, $>_i$ denotes the strict preference relation derived from $\gtrsim_i$.

Before any discussion of admissible price systems and trading strategies, some technical definitions are in order. Two adapted processes Y and Z are said to be versions of each other if $Y(t) = Z(t)$ a.s. for every $t \varepsilon [0,T]$. A martingale is an adapted stochastic process $Z = \{Z(t) ; t \varepsilon \underline{T}\}$ such that $Z(t)$ is integrable and $E(Z(t)|\mathcal{F}_s) = Z(s)$ a.s. for $0 \leqslant s \leqslant t \leqslant T$.[5] When $\mathbb{F}$ is right-continuous, any adapted martingale has a version whose sample paths are almost surely right-continuous with finite left-hand-limits (RCLL). (See Theorem 4 in Chapter VI of Meyer [34].) Since $\mathbb{F}$ has already been assumed to be right-continuous, we can therefore take an RCLL version of a martingale as a reference

point. Thus, an adapted martingale will always mean an RCLL process from now on.

Meyer [34, p. 291] states that an adapted RCLL process M is a local martingale if there exists an increasing sequence of stopping times $\{T_n\}$ such that

$$\lim_{n\to\infty} P(T_n = T) = 1$$

and the stopped process $\{M(t\wedge T_n)\ ;\ t\in \underline{T}\}$ is a martingale for each n.

A process $A = \{A(t), t\in\underline{T}\}$ is said to be in the class VF (for _variation finie_), or simply a VF process, if it is adapted, RCLL, and has sample paths of finite variation. (See Meyer [34], p. 249.) A process Z is called a semimartingale if it admits a decomposition $Z = M + A$, where M is a local martingale and A is a VF process. The decomposition may not be unique. (See Meyer [34], p. 298.)

The Borel field on $\Omega \times [0,T]$ generated by left-continuous processes adapted to $\mathbb{F}$ is called the predictable Borel field. (See Dellacherie and Meyer [12], p. 121.) A process H is said to be predictable if it is measurable with respect to the predictable Borel field. A predictable process H is locally bounded if $\sup_{0\le t\le T}|H(t)| < \infty$ a.s. (See Dellacherie [11].) It is known that an adapted process which is left-continuous with right limits (LCRL) is both predictable and locally bounded. The stochastic integral

$$\int H dZ$$

is well-defined if H is predictable and locally bounded and Z is a semimartingale. (For a discussion of this kind of integral, see the definitive treatise by Meyer [34].) Furthermore, if we put $Y(t) = \int_{(0,t]} H dZ$ then Y is also a semimartingale. Let Z and Y be two semimartingales, then the following integrals are well-defined:

$$\int_{(0,t]} Z(s-)dY(s) \quad \text{and} \quad \int_{(0,t]} Y(s-)dZ(s)$$

since the two processes $Z_- = \{Z(t-)\ ;\ t\in\underline{T}\}$ and $Y_- = \{Y(t-)\ ;\ t\in\underline{T}\}$ are LCRL, where by convention $Z(0-) = Z(0)$ and $Y(0-) = Y(0)$. We can therefore define the new

process:

$$[Z,Y]_t = Z(t)Y(t) - \int_{(0,t]} Z(s-)dY(s) - \int_{(0,t]} Y(s-)dZ(s). \qquad (2.4)$$

The process $[Z,Y]_t$ is called the joint variation of Z and Y and is a VF process. (See Meyer [34], p. 267.) It follows then that the product of two semimartingales is a semimartingale. Eq. (2.4) also gives the chain rule for differentiation:

$$d(Y(t)Z(t)) = Y(t-)dZ(t) + Z(t-)dY(t) + d[Y,Z]_t \quad . \qquad (2.5)$$

When $Z = Y$ , $[Z,Y] = [Y,Y]$ is the quadratic variation of Y. (Eq. (2.5) is simply the natural extension of Ito's formula.)

Suppose now that $Y = Y(0) + \int HdU$ , where H is predictable and locally bounded and U is a semimartingale, then Y, as mentioned above, is a semimartingale and

$$dY(t) = H(t)dU(t) \ . \qquad (2.6)$$

In this case we further have:

$$[Y,Z]_t = \int_{(0,t]} H(s)d[U,Z]_s \ , \qquad (2.7)$$

or equivalently,

$$d[Y,Z]_t = H(t)d[U,Z]_t \ , \qquad (2.8)$$

when Z is a semimartingale. (See Dellacherie [11].)

Returning to economics, an admissible price system for the traded consumption claims is an (N+1)-vector of essentially bounded semimartingales $S = \{S_n(t), n=0,1,2,\ldots,N;\ t\varepsilon\underline{T}\}$, with $S_0$ being bounded away from zero everywhere, where $S_n(t)$ denotes the price for claim n at time $t\varepsilon\underline{T}$. We restrict our attention to equilibrium price systems which are semimartingales since this class of processes admits the most general definition of a stochastic integral with integrand being predictable and locally bounded, and it contains many processes that have been chosen to model behaviors of asset prices: certain diffusion processes, the poisson jump processes, and mixtures of these two. The assumption that the admissible price processes for the $0^{th}$ claim are constrained to be bounded away

from zero can be rationalized as follows. In our economy, the payoff $d_0$ of the $0^{th}$ claim is bounded away from zero. If the price of a security at any time reflects its marginal contribution to agents' utilities, as it should, then as long as the marginal "social" valuation at time t of one unit of the consumption commodity to be delivered at T is bounded away from zero, we would naturally expect that the price of the $0^{th}$ claim at t is bounded away from zero. That this marginal social valuation is bounded away from zero is also natural in an economy where agents are not satiated at finite amount of consumption.

A trading strategy is defined to be an (N+1)-vector adapted process, $\theta = \{\theta_n(t)\,;\, n=0,1,\dots,N\,;\, t\in\underline{T}\}$, which is predictable and locally bounded. Given an admissible price system S, a trading strategy is said to be self-financing if

$$\theta(t)\cdot S(t) = \theta(0)\cdot S(0) + \int_{(0,t]} \theta(s)\cdot dS(s) \quad \text{a.s. } \forall t\in\underline{T}\,. \tag{2.9}$$

(This stochastic integral is well-defined since $\theta$ is predictable and locally bounded and S is a vector semimartingale. The symbol $\cdot$ denotes inner product.) This means that the difference of the value of $\theta$ at time t, i.e. $\theta(t)\cdot S(t)$, and its initial value $\theta(0)\cdot S(0)$, grows totally out of the capital gains:

$$\int_{(0,t]} \theta(s)\cdot dS(s)\ .$$

To put it another way, after the initial investment, for the strategy $\theta$ there is no new investment into and no withdrawal of funds out of the portfolio. (Harrison and Pliska [21] were the first to discuss this kind of trading strategy.) Now let $\Theta$ denote the set of self-financing trading strategies having the following property: $\forall\,\theta\in\Theta$,

$$\sup_{n=0,\dots,N} \left\{ \operatorname*{ess\,sup}_{\omega,t} |\theta_n(\omega,t)| \right\} < \infty\ . \tag{2.10}$$

This serves to rule out doubling strategies, since (2.10) also implies that

$$\operatorname*{ess\,sup}_{\omega,t} |\theta(\omega,t) \cdot S(\omega,t)| < \infty , \tag{2.11}$$

by the essential boundedness of S. For doubling strategies to be implementable, $\theta(t) \cdot S(t)$ cannot be bounded over $t \varepsilon \underline{T}$ and $\omega \varepsilon \Omega$. (For this point, see Harrison and Kreps [20].) It should also be clear that $\Theta$ is a linear space.

An <u>Equilibrium of Plans, Prices, and Price Expectations</u> (Radner [35]) is characterized by a price $\underline{a}$ for the unit consumption commodity at time 0, an admissible price system S for traded consumption claims, and I admissible trading strategies $\{\theta^i\}_{i=1}^{I}$, with $\theta^i \varepsilon \Theta$, one for each agent, such that $(-\theta^i(0) \cdot S(0)/a, \theta^i(T) \cdot d)$ is $\gtrsim_i$-maximal in the set $\{(r,x) \varepsilon V : (r,x) = (-\theta(0) \cdot S(0)/a, \theta(T) \cdot d)$ for some $\theta \varepsilon \Theta\}$, and $\sum_{i=1}^{I} \theta^i(t) = 0 \; \forall \varepsilon \underline{T}$ a.s., where d denotes the vector $(d_n)_{n=0}^{N}$.

The existence of an equilibrium is not an issue to be addressed in this paper. The main focus of this paper is to characterize some properties of the equilibrium price system if an equilibrium exists. (A special case of the economy we are dealing with will, however, be introduced in Section 4, and the existence of an equilibrium for it will be established.)

## 3. <u>Equilibrium and the Existence of an Equivalent Martingale Measure</u>

In carrying out the analysis in this section, familiarity with Harrison and Kreps [20] is assumed. (Henceforth, this is referred to as H&K.) The results to be shown are essentially extensions of theirs in the following direction. In their paper, the consumption space at time T is the space of square-integrable random variables. Agents' preferences are assumed to be continuous in the product topology on $R \times L^2(\Omega,\mathcal{F},P)$ generated by the Euclidean topology on R and the $L^2$-norm topology on $L^2(\Omega,\mathcal{F},P)$. And the trading strategies that are allowed there are simple trading strategies in the sense that agents can only trade at a finite

number of prespecified dates. In this section, consumption space for agents at time T is the space of essentially bounded random variables. Agents' preferences are assumed to be continuous in the product topology generated by the $L^1$-Mackey topology on X and the Euclidean topology on R. With these more restrictive assumptions on the consumption space and a different sense of continuity of agents' preferences, the results of H&K are rederived while allowing for admissible trading strategies to be predictable and essentially bounded encompassing simple trading strategies. Note that a predictable and essentially bounded trading strategy can trade continuously in [0,T].

First fix an admissible price system S. A claim $x \in X$ is said to be marketed at time zero if there exists a $\theta \in \Theta$ such that $\theta(T) \cdot d = x$ a.s. In this case, we say $\theta$ generates x. Let M denote the space of marketed claims at time zero. By the linearity of the stochastic integral and the fact that $\Theta$ is a linear space, it is clear that M is a linear subspace of X. A current price system is a linear functional $\pi : M \to R$. Let $x \in M$ and let $\theta$ be the strategy that generates x. Define $\pi(x) = \theta(0) \cdot S(0)$. Then $\pi$ gives the price at time zero of marketed claims. Now let $\Psi$ denote the set of all $L^1$-Mackey continuous and strictly positive linear functionals on X. A linear functional $\psi: X \to R$ is said to be strictly positive if $\psi(x) > 0$ for all $x \in X_+$, $x \neq 0$.

Now we define a vector stochastic process $S^* = \{S^*(t) = S(t)/S_0(t) ; t \in \underline{T}\}$. The next proposition shows that $S^*$ is an admissible price system. A result similar to Ito's Lemma is needed.

<u>Lemma 3.1</u>: Suppose Z is a real-valued semimartingale and F is a twice continuously differentiable function on R. Then the process F(Z) is also a semimartingale.

<u>Proof</u>: See Meyer [34], page 301.

Q.E.D.

Proposition 3.1: $S^*$ is an admissible price system.

Proof: First, we want to verify that $S^*$ is essentially bounded. This follows from the fact that S is essentially bounded and $S_0$ lies in a compact subinterval of $(0,\infty)$. Secondly, we claim that $S^*$ is a vector semimartingale. This follows since, by Lemma 3.1, $1/S_0$ is a semimartingale and the product of two semimartingales is a semimartingale (see (2.4)). Hence $S^*$ is an admissible price system. Q.E.D.

Denoting the set of self-financing trading strategies with respect to $S^*$ by $\Theta^*$, we have the following characterization:

Proposition 3.2: $\theta\varepsilon\Theta$ if and only if $\theta\varepsilon\Theta^*$, where $\Theta^*$ is defined as those predictable and essentially bounded adapted processes $\theta$ such that

$$\theta(t)\cdot S^*(t) = \theta(0)\cdot S^*(0) + \int_{(0,t]} \theta(s)\ \ dS^*(s), \text{ a.s. } \forall t\varepsilon \underline{T} .$$

Proof: Let $\theta\varepsilon\Theta$. Then by definition of a self-financing trading strategy with respect to S we have:

$$\theta(t)\cdot S(t) = \theta(0)\cdot S(0) + \int_{(0,t]} \theta(s)\cdot dS(s) \ ,$$

or equivalently,

$$d(\theta(t)\cdot S(t)) = \theta(t)\cdot dS(t) \ .$$

We want to show that $d(\theta(t)\cdot S^*(t)) = \theta(t)\cdot dS^*(t)$. Putting $\beta(t) = 1/S_0(t)$ we have

$$\begin{aligned} dS^*(t) &= d(S(t)\beta(t)) \\ &= S(t-)d\beta(t) + \beta(t-)dS(t) + d[S,\beta]_t \ . \end{aligned}$$

The second line follows from (2.5). By the same reasoning

$$\begin{aligned} d(\theta(t)\cdot S^*(t)) &= d(\theta(t)\cdot S(t)\beta(t)) \\ &= \theta(t)\cdot S(t-)d\beta(t) + \beta(t-)d(\theta(t)\cdot S(t)) + d[\beta,\theta\cdot S]_t \\ &= \theta(t)\cdot S(t-)d\beta(t) + \theta(t)\cdot\beta(t-)dS(t) + \theta(t)\cdot d[\beta,S]_t \\ &= \theta(t)\cdot [S(t-)d\beta(t) + \beta(t-)dS(t) + d[\beta,S]_t] \\ &= \theta(t)\cdot d(S(t)\beta(t)) = \theta(t)\cdot dS^*(t) \ . \end{aligned}$$

The second line follows from the proof of Proposition 3.24 in Harrison and Pliska [21]; the third line follows from (2.6) and (2.8).

The converse is virtually identical, so we omit it.

Q.E.D.

Given the admissible price system $S^*$, a claim $x \in X$ is said to be marketed at time zero if there exists $\theta \in \Theta^*$ such that $\theta(T) \cdot d = x$ a.s. Let $M^*$ denote the space of marketed claims given $S^*$. It is clearly a linear space and we have the following characterization:

Corollary 3.2: $x \in M$ if and only if $x \in M^*$.

Proof: Let $x \in M$. There exists $\theta \in \Theta$ such that $\theta(T) \cdot d = x$ a.s. From Proposition 3.2, we know $\theta \in \Theta^*$, and therefore $x \in M^*$. The converse is identical.

Q.E.D.

Now we are ready to state the following:

Proposition 3.3: If $\{(S,a), (\theta^i)_{i=1}^I\}$ is an equilibrium then $\{(S^*, a/S_0(0)), (\theta^i)_{i=1}^I\}$ is an equilibrium. Conversely, given $S_0$ lying in a compact subinterval of $(0,\infty)$, if $\{(S^*,a^*), (\theta^i)_{i=1}^I\}$ is an equilibrium then $\{(S,a), (\theta^i)_{i=1}^I\}$ is an equilibrium with $S(t) = S^*(t)S_0(t)$ $\forall\, t \in [0,T]$ a.s. and $a = a^* S_0(0)$.

Proof: Suppose that $\{(S,a), (\theta^i)_{i=1}^I\}$ is an equilibrium. We claim that $\{(S^*, a/S_0(0)), (\theta^i)_{i=1}^I\}$ is also an equilibrium. Suppose the contrary. Since $(\theta^i)_{i=1}^I$ are market-clearing, there must exist some $i$ and some $\theta \in \Theta^*$ such that

$$(-\theta(0) \cdot S^*(0)S_0(0)/a\, ,\, \theta(T) \cdot d) >_i (-\theta^i(0) \cdot S(0)/a\, ,\, \theta^i(T) \cdot d) \quad .$$

Recall that $S^*(0) = S(0)/S_0(0)$. We thus have

$$(-\theta(0) \cdot S(0)/a\, ,\, \theta(T) \cdot d) >_i (\theta^i(0) \cdot S(0)/a\, ,\, \theta^i(T) \cdot d) \quad . \tag{3.1}$$

By Proposition 3.2, we know $\theta \in \Theta$, thus (3.1) is a contradiction to the fact that $\{(S,a), (\theta^i)_{i=1}^I\}$ is an equilibrium. So $\{(S^*, a/S_0(0)), (\theta^i)_{i=1}^I\}$ is an equilibrium. The converse is identical.

Q.E.D.

Now let S be an equilibrium price system, with $\pi$: $M \to R$ the current price system associated with it. By Proposition 3.3, $S^*$ is also an equilibrium price system. It is clear that the current price system associated with $S^*$, $\pi^*$: $M \to R$, is such that $\pi^*(m) = \pi(m)/\pi(d_0)$ for all $m \varepsilon M$. Theorem 1 of H&K shows that there exist extensions of $\pi$ and $\pi^*$ to all of X that lie in $\Psi$. Denoting such extensions of $\pi$ and $\pi^*$ by $\psi$ and $\psi^*$, respectively.

<u>Proposition 3.4</u>: $\psi^*$ is an extension of $\pi^*$ from M to all of X that lies in $\Psi$ if and only if $\psi$ is an extension of $\pi$ from M to all of X that lies in $\Psi$, where $\psi$ and $\psi^*$ are linked by

$$\psi^*(x) = \psi(x)/\pi(d_0) \ .$$

<u>Proof</u>: Consider the if part first. Let $\psi$ be an extension of $\pi$ from M to all of X and $\psi \varepsilon \Psi$. Define $\psi^*$: $X \to R$ by $\psi^*(x) = \psi(x)/\pi(d_0)$. Since $\pi(d_0) = S_0(0) > 0$, it follows that $\psi^*$ is a strictly positive $L^1$-Mackey continuous linear functional. We are left to show that $\psi^*$ is an extension of $\pi^*$. Let $m \varepsilon M$. Then

$$\psi^*(m) = \frac{\psi(m)}{\pi(d_0)} = \frac{\pi(m)}{\pi(d_0)} = \pi^*(m) \ .$$

The first equality follows from the definition of $\psi^*$; the second from the extension property of $\psi$; and the third from the definition of $\pi^*$. Therefore, $\psi^*$ is an extension of $\pi^*$ that lies in $\Psi$.

The arguments for proving the only if part are similar, so we omit them.

Q.E.D.

The above series of propositions and corollaries have established that changing the numeraire in our economy is purely economically neutral as long as some "boundedness" condition is satisfied. This is hardly a surprise. In an economic equilibrium only relative prices are determined, we can therefore normalize prices by a convenient numeraire. The boundedness of $S_0$ is required to preserve the admissibility of the price system.

In the next proposition, a property of $S^*$ will be characterized. This proerty is important later in the development of this paper. Defining

$$d^* \equiv d/d_0 \quad ,$$

we have

Proposition 3.5: If S is an equilibrium price system for traded claims, then

$$S^*(\omega,T) = d^*(\omega) \quad \text{a.s.}$$

Proof: Since S is an equilibrium price system for traded claims, there must exist an $\alpha \varepsilon X_+$ such that

$$S(\omega,T) = \alpha(\omega)d(\omega) \quad \text{a.s.}$$

Readers should convince themselves that the existence of such an $\alpha$ follows from the equilibrium property of S. Therefore we have

$$S^*(\omega,T) = \alpha(\omega)d(\omega)/(\alpha(\omega)d_0(\omega)) = d^*(\omega) \quad \text{a.s.}$$

Q.E.D.

For notational simplicity, in the rest of this paper, we shall fix an equilibrium price system S for traded claims and assume that S has been normalized such that $S_0(t) = 1 \quad \forall t \varepsilon [0,T]$. From Proposition 3.5 we then know $S(T) = d^*$ a.s.

Speaking with reference to the probability measure P, an equivalent martingale measure is a probability measure on $(\Omega,\mathcal{F})$ which has the following properties. First, P and Q are equivalent in the probability sense, meaning that $P(B) = 0$ if and only if $Q(B) = 0$, for all $B \varepsilon \mathcal{F}$. (It follows from the Radon-Nikodym theorem and the fact that Q is a probability measure that the Radon-Nikodym derivative $\rho = dQ/dP$ is integrable, i.e. $\rho \varepsilon L^1(\Omega,\mathcal{F},P)$, and has unit expectation.) Secondly, the price system S is a vector bounded martingale under Q. That is, denoting the expectation operator under Q by $E^*(\cdot)$, we have

$$E^*[\, S(s)|\mathcal{F}_t] = S(t) \text{ Q-a.s. } \forall s,t \varepsilon [0,T] \text{ and } t \leqslant s.$$

Before going to the main result of this section, we state a property of stochastic integration which applies when the integrator is a square-integrable martingale.

Let Z be an adapted square-integrable martingale on $(\Omega,\mathcal{F},P)$ and H a predictable and locally bounded process. The process $[Z,Z]$ is well defined since a martingale is a semimartingale. Furthermore, we have (see Meyer [34], p. 267.):

$$E\left[\int_{(0,T]} d[Z,Z]_t\right] < \infty \; . \tag{3.2}$$

The integral $\int H d[Z,Z]$ is defined path-by-path in the Lebesgue-Stieltjes sense.

Lemma 3.2: If we put $Y(t) = H(0)Z(0) + \int_{(0,T]} H(s)dZ(s)$, then Y is a square-integrable martingale on $(\Omega,\mathcal{F},P)$ if

$$E\left[\int_{(0,T]} H^2(s)d[Z,Z]_s\right] < \infty \; .$$

Proof: See Dellacherie [11], page 20.

Q.E.D.

Note that if H is bounded, i.e.

$$\operatorname*{ess\,sup}_{\omega,t} |H(\omega,t)| < \infty \quad ,$$

then

$$E\left[\int_{(0,T]} H^2(t)d[Z,Z]_t\right] \leq \operatorname*{ess\,sup}_{\omega,t} (|H(\omega,t)|)^2 \, E\left[\int_{(0,T]} d[Z,Z]_t\right] < \infty \; .$$

So if $\theta \varepsilon \Theta$ and S is a vector martingale, then $\theta \cdot S$ is also a square-integrable (in fact, a bounded) martingale.

Theorem 3.1: Let S be an equilibrium price system and $\pi$ the current price system associated with it. Then there exists a one-to-one mapping between equivalent martingale measures Q and linear functionals $\psi \varepsilon \Psi$ such that $\psi$ is an extension of $\pi$ to all of X. This correspondence is given by

$$Q(B) = \psi(1_B d_0) \quad \forall B \varepsilon \mathcal{F} \quad \text{and} \quad \psi(x) = E^*(x/d_0) \quad \forall x \varepsilon X.$$

Proof: We note first that a linear functional $\psi$ on X is $L^1$-Mackey continuous if and only if, for all $x\varepsilon X$, $\psi(x) = E(x\rho)$ for some $\rho\varepsilon L^1(\Omega,\mathcal{F},P)$. (This follows since the $L^1$-Mackey topology is the strongest topology on X such that its topological dual is $L^1(\Omega,\mathcal{F},P)$.) Now suppose that $\psi\varepsilon\Psi$ is an extension of $\pi$. Since $\psi$ is strictly positive and $L^1$-Mackey continuous, there exists $\rho\varepsilon L^1(\Omega,\mathcal{F},P)$ with $P\{\rho > 0\} = 1$ such that $\psi(x) = E(\rho x)\ \forall x\varepsilon X$. Define a set function

$$Q(A) = \int_A \rho(\omega)d_0(\omega)P(d\omega)\ , \qquad A\varepsilon\mathcal{F}.$$

Since $d_0\varepsilon X_{++}$ and $Q(\Omega) = E(\rho d_0) = \psi(d_0) = 1$, by Radon-Nikodym theorem it is clear that Q is a probability measure on $(\Omega,\mathcal{F})$ equivalent to P. Next we want to show that under Q, S is a vector bounded martingale.

Fix an integer k, $0 < k \leq N+1$. Let $0 \leq t_1 \leq t_2 \leq T$ and $B\varepsilon\mathcal{F}_{t_1}$. Consider the following trading strategy:

$$\begin{aligned}\theta_k(\omega,t) &= 1 \quad \text{for} \quad t\varepsilon(t_1,t_2] \text{ and } \omega\varepsilon B\\ &= 0 \quad \text{otherwise}\ ,\\ \theta_0(\omega,t) &= -\ S_k(\omega,t_1) \text{ for } t\varepsilon(t_1,t_2] \text{ and } \omega\varepsilon B\\ &= S_k(\omega,t_2) - S_k(\omega,t_1) \text{ for } t\varepsilon(t_2,T] \text{ and } \omega\varepsilon B\\ &= 0 \quad \text{otherwise}\ ,\end{aligned}$$

$$\text{and} \quad \theta_n(\omega,t) = 0 \quad \text{for all } n \neq 0,k\ .$$

We claim that $\theta\varepsilon\Theta$. Firstly, $\theta$ is bounded since S is. Secondly, $\theta$ is predictable because $\theta$ is left-continuous. Lastly, we want to verify that $\theta$ is self-financing. At $t_1$, the value of $\theta$ is zero. For $\omega\varepsilon B$, the value of $\theta$ at $t_1+$ is $S_k(\omega,t_1+) - S_k(\omega,t_1)$. Recall that the equilibrium price system S is RCLL P-a.s. Therefore, the value of $\theta$ at $t_1+$ is zero almost surely. From then on through time $t_2$, $\theta$ is obviously self-financing. The fact that it is self-financing after time $t_2$ also follows from the right continuity of S. Thus we have shown that $\theta\varepsilon\Theta$.

The payoff of $\theta$ at T in units of the consumption commodity is $1_B(S_k(t_2) - S_k(t_1)) \cdot d_0$. This claim is marketed and has a zero price at time zero. Hence $\psi(1_B(S_k(t_2) - S_k(t_1)) \cdot d_0) = \pi(1_B(S_k(t_2) - S_k(t_1))d_0) = 0$. Since $B \varepsilon \mathcal{F}_{t_1}$ is arbitrary, this implies that

$$E\,(1_B S_k(t_1) d_0 \rho) = E\,(1_B S_k(t_2) d_0 \rho) \quad \forall B \varepsilon \mathcal{F}_{t_1} .$$

Equivalently,

$$\int_B S_k(t_1) Q(d\omega) = \int_B S_k(t_2) Q(d\omega) \quad \forall B \varepsilon \mathcal{F}_{t_1} .$$

This simply says that $S_k$ is a martingale under Q. Since k is arbitrary, we have therefore shown that S is a vector martingale under Q. The boundedness follows from the fact that S is bounded under P, and that P and Q are equivalent.

Conversely, let Q be an equivalent martingale measure. Putting $\rho = (dQ/dP)/d_0$, $\rho \varepsilon L^1(\Omega, \mathcal{F}, P)$ by the definition of an equivalent martingale measure and the fact that $d_0 \varepsilon X_{++}$. We know $\rho$ is strictly positive P-a.s. and Q-a.s., since P and Q are equivalent. Define an $L^1$-Mackey continuous and strictly positve linear functional $\psi$ by

$$\psi(x) = E(x\rho) = E^*(x/d_0) \quad x \varepsilon X.$$

We want to show that $\psi$ is an extension of $\pi$. Let $\theta$ be any trading strategy in $\Theta$, and $m = \theta(T) \cdot d$. Then $m \varepsilon M$. By construction

$$\begin{aligned} \psi(m) = E^*(m/d_0) &= E^*(\theta(T) \cdot d^*) \\ &= E^*(\theta(T) \cdot S(T)). \end{aligned}$$

The second line follows from Proposition 3.5. Again, by the definition of an equivalent martingale measure, we know that S is a vector bounded martingale under Q and therefore a vector square-integrable martingale under Q. Now from Lemma 3.2, the discussion after it, as well as the fact that the predictability and essential boundedness of a process are preserved under a substitution of an equivalent probability measure, we get

$$\psi(m) = E^*(\theta(T) \cdot S(T)) = \theta(0) \cdot S(0) = \pi(m) .$$

Hence $\psi$ is an extension of $\pi$ that lies in $\Psi$.

Q.E.D.

This theorem is the continuous analogue of H&K's Theorem 2. Fix an equilibrium price system, S. A probability measure Q equivalent to P can then be constructed such that S is a vector martingale under Q. It will be shown in the next section that this result has important consequences in the sense that many properties of a martingale that are invariant under the substitution of an equivalent probability measure can be used to characterize the behavior of equilibrium asset prices.

## 4. The Continuous Information Structure and the Equilibrium Price System

In this section, a definition for a continuous information structure is given, which is originally due to Harrison [19]. It will be shown that if the information structure for agents in our economy is continuous, then the equilibrium price system must have continuous sample paths. Further, if the filtration is not trivial throughout, then the equilibrium price system not only has continuous sample paths but the sample paths must involve unbounded variation over a non-trivial subinterval of $[0,T]$ for those securities whose payoffs at time T are not constants P-a.s. in units of that of the $0^{th}$ claim. Before giving a formal definition of a continuous information structure, some remarks are in order. First, let $Y = \{Y(t), t\in \underline{T}\}$ and $Z = \{Z(t), t\in \underline{T}\}$ be two martingales on $(\Omega, \mathcal{F}, P)$ adapted to $\mathbb{F}$. If Z is a modification of Y, then $P\{Y(t) = Z(t), \forall\, t \in \underline{T}\} = 1$. This follows directly from the right-continuity of martingales. Here Y and Z are said to be indistinguishable. Note that if a martingale has a continuous modification then it is indistinguishable from a continuous process. Second, let B be any set in $\mathcal{F}$. Then $E(1_B | \mathcal{F}_t) = P\{B | \mathcal{F}_t\}$ is the posterior probability assessment at time t of event B. The process $\{P\{B|\mathcal{F}_t\} ; t\in \underline{T}\}$ is an adapted martingale on $(\Omega, \mathcal{F}, P)$, and therefore

has right-continuous sample paths P -a.s. (Recall that we always take an RCLL version of a martingale as a reference point. In the above statement we took an RCLL version of the conditional expectation.)

Definition: The information structure (already assumed to be right-continuous) $\mathbb{F} = \{\mathcal{F}_t, t\epsilon\underline{T}\}$ is said to be continuous if for every $B\epsilon\mathcal{F}$, the martingale $\{P(B|\mathcal{F}_t)\,;\, t\epsilon\underline{T}\}$ has a continuous modification.

This definition says that agents' posterior probability assessments of any evert $B\epsilon\mathcal{F}$ evolve continuously through time P-a.s., which should conform with any intuitive ideas of a continuous information structure. In the following proposition, it will be shown that this definition of a continuous information structure has an interesting consequence, wh...h will be used throughout the remainder of this paper.

Proposition 4.1: The information structure $\mathbb{F} = \{\mathcal{F}_t, t\epsilon\underline{T}\}$ is continuous, if and only if every martingale on $(\Omega,\mathcal{F},P)$ that is adapted to $\mathbb{F}$ has a continuous modification.

Proof: Since $\{P(B|\mathcal{F}_t)\}$ is a martingale for all $B\epsilon\mathcal{F}$, the if part is trivial.

For the only if part, we take cases.

Case 1. Let Y be a positive integrable random variable on $(\Omega,\mathcal{F},P)$. Then we can find a sequence of non-negative simple integrable random variables $\{Y_n\}$ such that $Y_n \leq Y$ P -a.s. and $\lim_{n\to\infty} Y_n = Y$ P -a.s. Now we claim that $Y_n - Y \to 0$ in $L^1$. To see this, we note that $|Y_n - Y| \leq 2|Y|$, and $E(Y) < \infty$. The latter follows from the hypothesis. Theorem 4.1.4 in Chung [6] says that convergence almost surely implies convergence in $L^1$ if the sequence of random variables is dominated by an integrable random variable. Hence, we have

$$E(|Y_n - Y|) \to 0 \quad \text{as } n \to \infty .$$

That is, $Y_n - Y \to 0$ in $L^1$. We thus can choose a sequence of simple integrable random variables $\{Y_n\}$ such that

$$E(|Y_n - Y|) < 1/n^2 , \quad n = 1,2,\ldots$$

Let $Y_n(t)$ be an RCLL version of $E(Y_n|\mathcal{F}_t)$. (Here we mean that for every $t\varepsilon\underline{T}$, let $Y_n(t)$ be a version of $E(Y_n|\mathcal{F}_t)$ such that $\{Y_n(t), t\varepsilon\underline{T}\}$ is an RCLL process.) Then $\{Y_n(t), \mathcal{F}_t ; t\varepsilon\underline{T}\}$ has a continuous modification. This is true since it is true when $Y_n$ is an indicator function, therefore true for simple random variables. Now let $Y(t)$ be an RCLL version of $E(Y|\mathcal{F}_t)$. We want to show that the martingale $\{Y(t), \mathcal{F}_t ; t\varepsilon\underline{T}\}$ has a continuous modification.

First we observe that $\{Y_n(t) - Y(t), \mathcal{F}_t ; t\varepsilon\underline{T}\}$ is a martingale. By a generalization of Kolmogorov's inequality (Theorem 1 in Chapter VI of Meyer [34]) we have:

$$P\left\{\sup_{0\leqslant t\leqslant T} | Y_n(t) - Y(t) | > \alpha\right\} \leqslant \frac{1}{\alpha} E(|Y - Y_n|) \leqslant \frac{1}{\alpha n^2} .$$

Consequently, by the Borel-Cantelli Lemma,

$$\sup_{0\leqslant t\leqslant T} | Y_n(t) - Y(t)| \to 0 \text{ as } n \to \infty .$$

Or equivalently, $Y_n(t)$ converges uniformly in t to $Y(t)$ with probability one. This implies that $Y(t)$ is continuous P - a.s.

<u>Case 2</u>. Let Y be any integrable random variable. Then we can write $Y = Y^+ - Y^-$, where $Y^+ = Y\vee 0$, and $Y^- = (-Y)\vee 0$. Now it is straightforward to apply the argument in Case 1 to $Y^+$ and $Y^-$ respectively. Then the proposition follows from the fact that the sum of two continuous functions is continuous. Q.E.D.

Proposition 4.1 shows that our definition for a continuous information structure is in fact equivalent to saying that $\mathbb{F}$ is continuous if any adapted martingale on $(\Omega,\mathcal{F},P)$ is indistinguishable from a continuous process.

Note that the above definition of a continuous information is cast on the probability space $(\Omega,\mathcal{F},P)$. Our intuition would suggest that the continuity of an information structure should mainly involve the fine structure of the filtration, and not the particular probability measure involved. The following proposition, which is originally due to Harrison [19], formalizes this basic idea, and shows that the definition of a continuous information structure is

invariant under the substitution of an equivalent probability measure. (Here we should note that in Harrison [19], after he gives the definition of a continuous information structure, he goes on to conjecture that Proposition 4.1 is correct. He sets the following proposition as a homework problem. My contribution is to provide a proof for Proposition 4.1.) For the completeness of this paper, a proof will be given, but before doing that, a lemma is recorded. Let Q be a probability measure on $(\Omega, \mathcal{F})$, which is equivalent to P with $dQ = \xi dP$, where $\xi$ is the Radon-Nikodym derivative of Q with respect to P. Of course $\xi$ is strictly positive except possibly on a P-null set, and is of unit expectation under P. Now let $\xi(t)$ be an RCLL version of $E[\xi|\mathcal{F}_t]$. Then $\{\xi(t), \mathcal{F}_t ; t \varepsilon \underline{T}\}$ is a strictly positive martingale on $(\Omega, \mathcal{F}, P)$.

<u>Lemma 4.1</u>: Let $Z = \{Z(t) ; t \varepsilon \underline{T}\}$ be a process adapted to $\mathbb{F}$. Then Z is a martingale on $(\Omega, \mathcal{F}, Q)$ if and only if $\{Z(t)\xi(t) ; t \varepsilon \underline{T}\}$ is a martingale on $(\Omega, \mathcal{F}, P)$.

<u>Proof</u>: See, for example, Gihman and Skorohod [16], p. 149. Q.E.D.

<u>Proposition 4.2</u>: The filtration $\mathbb{F}$ represents a continuous information structure on $(\Omega, \mathcal{F}, P)$, if and only if it is a continuous information structure on $(\Omega, \mathcal{F}, Q)$ for all Q equivalent to P.

<u>Proof</u>: First, let $\mathbb{F}$ be a continuous information structure on $(\Omega, \mathcal{F}, P)$ and Z be a martingale adapted to $\mathbb{F}$ under Q. We want to show that Z has a continuous modification. Or equivalently, except for a Q-null set, the sample functions of Z are continuous.

First note that $\{\xi(t), \mathcal{F}_t ; t \varepsilon \underline{T}\}$ is a martingale on $(\Omega, \mathcal{F}, P)$, so it has a continuous modification by Proposition 4.1 and the fact that $\mathbb{F}$ is continuous. From the right-continuity of a martingale, we know that there exists a P-null set $N_1$ such that

$$\forall \omega \varepsilon \Omega \setminus N_1, \quad \xi(\omega, t) \text{ is a continuous function of } t .$$

From Lemma 4.1, we know that if Z is a martingale on $(\Omega, \mathcal{F}, Q)$, then $\{Z(t)\xi(t), \mathcal{F}_t ; t \varepsilon \underline{T}\}$ is a martingale on $(\Omega, \mathcal{F}, P)$. Therefore, there exists a P-null

set $N_2$ such that

$$\forall \omega \varepsilon \Omega \backslash N_2 \ , \quad Z(\omega,t)\xi(\omega,t) \ \text{ is a continuous function of } t \ .$$

Now let $N_3$ be a P-null set such that

$$\forall \omega \varepsilon \Omega \backslash N_3 \ , \quad \xi(\omega,t) > 0 \quad \forall t \varepsilon \underline{T} \ .$$

Finally we define

$$\forall \omega \varepsilon \Omega \backslash (N_1 \cup N_2 \cup N_3), \ M(\omega,t) = Z(\omega,t) \ , \quad \forall t \varepsilon \underline{T} \ ,$$

and

$$\forall \omega \varepsilon \ N_1 \cup N_2 \cup N_3 \ , \ M(\omega,t) = 0 \ , \quad \forall t \varepsilon \underline{T} \ .$$

Then $\{M(t) \ ; \ t \varepsilon \underline{T}\}$ has continuous sample paths everywhere. By construction, $Z(t) = M(t)$, $\forall t \varepsilon \underline{T}$ except on a P-null set $N = N_1 \cup N_2 \cup N_3$. Therefore Z has continuous sample paths except on a set of P-measure zero. But since P and Q are equivalent probability measures, Z has continuous sample paths except on a set of Q-measure zero, which was to be shown.

The converse is identical, so we delete its proof. Q.E.D.

Recall from Section 3 that if S is an equilibrium price system, then there exists a probability mreasure Q equivalent to P such that S is a vector adapted martingale under Q.

The main result of this section is:

Theorem 4.1: If the information structure of agents is continuous, then the equilibrium price system has a continuous modification.

Proof: Recall that S is numerated such that $S_0(t) = 1$, $\forall t \varepsilon \underline{T}$ and S is a vector martingale under Q. By Proposition 4.2, $\mathbb{F}$ is a continuous information structure on $(\Omega, \mathcal{F}, Q)$, so that S has a continuous modification under Q by Proposition 4.1 Now, since P and Q are equivalent, S has a continuous modification under P. Q.E.D.

Theorem 4.1 says that if agents' information structure is continuous, then any equilibrium price system must have almost all of its sample paths continuous.

In an economy under uncertainty, if the equilibrium price system is of bounded variation, then all the traded securities are locally riskless,[6] since almost all the sample paths of S will be differentiable almost everywhere (with respect to the Lebesque measure) on [0,T]. That is, roughly, at almost every time $t \in \underline{T}$ we know the direction of the price movement in the next instant. We can in fact say a little bit more about the sample paths of an equilibrium price system by first recording a fact first discovered by Fisk [15].

<u>Lemma 4.2</u>: Let $Z = \{Z(t), \mathcal{F}_t ; t \in \underline{T}\}$ be a continuous martingale on $(\Omega, \mathcal{F}, Q)$. Then either the sample paths of Z are of unbounded variation over some subinterval of [0,T] with strictly positive probability or Z is a constant throughout. Equivalently, on any subinterval of $\underline{T}$, either the paths of Z are of unbounded variation or they are constants on the subinterval.

<u>Theorem 4.2</u>: If $\mathcal{F}$ is not trivial, and if the information structure $\mathbb{F}$ is continuous, then the equilibrium prices for those claims whose payoffs at time T are not constants in units of the $0^{th}$ claim, will have sample paths of unbounded variation over some sub-interval of [0,T] with strictly positive probability.

<u>Proof</u>: Consider the prices for, say, claim n. Suppose that there exists two sets $A, B \in \mathcal{F}$, with strictly positive P-measure such that

$$d_n^*(\omega) \neq d_n^*(\omega') \ , \quad \forall \omega \in A, \text{ and } \omega' \in B \ ,$$

that is, $d_n^*$ is not a constant P-a.s., where $d_n^* \equiv d_n/d_0$. We claim that $S_n$ has sample paths of unbounded variation over some sub-interval of [0,T] with strictly positive probability. Suppose this is not the case. Then by Lemma 4.2, $S_n(t)$ must be a constant for every $t \in \underline{T}$:

$$S_n(t) = E^*[\, d_n^* | \mathcal{F}_t \,] = E^*[\, d_n^* | \mathcal{F}_T \,] = S_n(T) \quad \text{Q-a.s.}$$

In the above expression, we have used the fact that $d^* = S(T)$ P - a.s. and therefore Q - a.s., and that S is a vector martingale under Q. Recall the assumption the $\mathcal{F} = \mathcal{F}_T$. We have $E(d_n^*|\mathcal{F}_T) = d_n^*$ Q - a.s., and thus

$$E^*[d_n^*|\mathcal{F}_t] = d_n^*, \quad Q - \text{a.s.} \quad \forall t \in \underline{T}.$$

In particular, for t = 0:

$$E^*[d_n^*] = d_n^*, \quad Q - \text{a.s.}$$

This implies that $d_n^*$ is a constant Q - a.s. But since P and Q are equivalent probability measures, $d_n^*$ is a constant P-a.s. This contradicts the assumption that $d_n^*$ is not a constant P-a.s. Therefore we have proved the claim that $S_n$ has sample paths of unbounded variation over some sub-interval of [0,T] with strictly positive probability. Since the above argument holds for all of the claims, we are done. Q.E.D.

Combining Theorems 4.1 and 4.2, we have an interesting characterization of equilibrium price systems. The continuity of the sample functions of the equilibrium asset prices derives from the continuity of agents' information structure. The continuity of agents' information structure also requires that the equilibrium asset prices have sample functions of unbounded variation on some non-trivial subinterval of [0,T] with strictly positive probability except for the $0^{th}$ claim and assets having payoff structures that are proportional P-almost surely to that of the $0^{th}$ claim. One natural question that remains to be addressed is whether the continuity of the equilibrium price system also implies the continuity of agents' information structure. The answer is clearly no. For example, if all the claims traded in the economy have proportional payoffs, then no matter what the information structure is, the equilibrium price system must have a continuous modification. A partially converse statement of Theorem 4.1 is, however available. We first give a definition of market completeness, which is a natural extension from the Arrow-Debreu framework.

Definition: An equilibrium price system S is said to yield complete markets if for every claim $x \in X$, there exists a self-financing trading stretegy $\theta \in \Theta$ such that $\theta(T) \cdot d = x$ almost surely.

In essence, a complete market implies that $M = X$, that is, every random variable $x \in X$ is marketed at time zero. (For discussion of complete markets, see, for example, H&K, Harrison and Pliska [21], and Duffie and Huang [13].)

Theorem 4.3: If the equilibrium price system S yields complete markets and has a continuous modification, then the information structure $\mathbb{F}$ is continuous.

Proof: Let B be any set in $\mathcal{F}$. By the hypothesis that S yields complete markets we know there exists $\theta \in \Theta$ such that $\theta(T) \cdot d = 1_B \cdot d_0$ a.s. This is equivalent to $\Theta(T) \cdot S(T) = 1_B$ a.s. (Recall Proposition 3.5.) It is easy to see that the value of this claim, $1_B \cdot d_0$, at time $t \in \underline{T}$ must be equal to $\theta(t) \cdot S(t)$ almost surely. Let $Y(t)$ denote the price at time t of this claim, then

$$Y(t) = \theta(0) \cdot S(0) + \int_{(0,t]} \theta(s) \cdot dS(s) \quad \text{a.s.}$$

Since S has a continuous modification, Y has a continuous modification under P. This fact follows from Theorem 20 in Chapter IV of Meyer [34].

Next, it follows from Lemma 3.2 that Y is a martingale under Q, that is,

$$Y(t) = E^*[\, Y(T) | \mathcal{F}_t] = E^*[\, \theta(T) \cdot S(T) | \mathcal{F}_t \,]$$
$$= E^*[\, 1_B | \mathcal{F}_t \,] \qquad \text{Q-a.s.}$$

Note that Y also has a continuous modification under Q, since P and Q are equivalent. Therefore, the martingale $\{E^*[\, 1_B | \mathcal{F}_t], \mathcal{F}_t \,;\, t \in \underline{T}\}$ on $(\Omega, \mathcal{F}, Q)$ has a continuous modification.

Since B is an arbitrary set in $\mathcal{F}$, we have thus shown that $\mathbb{F}$ represents a continuous information structure on $(\Omega, \mathcal{F}, Q)$. By Proposition 4.2, $\mathbb{F}$ is also a continuous information structure on $(\Omega, \mathcal{F}, P)$, which was to be shown.

Q.E.D.

In this section we have shown that if the information structure is continuous then the equilibrium price system S must have continuous sample paths. Except for uninteresting cases, $S_n$ must be of unbounded variation over a non-trivial subinterval of [0,T] with strictly positive probability for n=1,2,...,N. Conversely, if the markets are complete, and if S is continuous, then the information structure has to be continuous.

Remark: It should be noted that all the results in this section will remain valid if agents are endowed with different probability measures on $(\Omega,\mathcal{F})$ as long as those different probability measures are all equivalent to one another.

## 5. A Canonical Example

The properties of equilibrium asset prices described in the previous section are, of course, provisional on the existence of an equilibrium. Although proving the existence of an equilibrium in a general multi-agent setting is beyond the scope of this paper, in this section, an autarchy example of the economy will be provided and the existence of an equilibrium will be established. Furthermore, it will be shown that if the information structure is generated by a standard Brownian motion, then the equilibrium prices for traded consumption claims can be represented as Ito integrals. This Ito-integral-representation property of equilibrium asset prices has been the primitive assumption for many continuous-time intertemporal equilibrium models of financial economics. (See, for example, Merton [30], Breeden [4].) It should also be clear in the sequel that the above-mentioned representation property for asset prices is valid in a multi-agent economy if an equilibrium indeed exists. That is, the representation as an Ito integral only requires that the filtration be generated by a Brownian motion. (See the remark following (5.2.4).)

### 5.1 The Brownian Motion Filtration is a Continuous Information Structure

Let $\{w(t)\,;\,t\varepsilon\underline{T}\}$ be a standard Brownian motion with continuous sample paths defined on $(\Omega,\mathcal{F},P)$ generating a filtration $\mathbb{F}^w = \{\mathcal{F}_t^w\,;\,t\varepsilon\underline{T}\}$. (We assume that $\mathcal{F}_0^w$ contains all the P-negligible sets of $\mathcal{F}$ and $\mathcal{F}_T^w = \mathcal{F}$.) The following conditions are sufficient for the existence of the stochastic integral $\int_0^T \phi(t)dw(t)$. (See Chapter 4 of Lipster and Shiryayev [28].)

1. $\phi$ is adapted to $F^w$ ,
2. $\int_0^T \phi(t)^2 dt < \infty$ a.s.

Let $\Phi$ denote the family of functions satisfying the above conditions. An important result, due to Clark [7, Theorem 3] is:

Lemma 5.1.1: Let $\eta$ be an integrable random variable defined on $(\Omega,\mathcal{F},P)$. There exists a function $\phi\varepsilon\Phi$ such that for all $s,t, 0\leqslant s\leqslant t\leqslant T$ ,

$$E[\eta|\mathcal{F}_t^w] = E[\eta] + \int_0^t \phi(u)dw(u) \quad \text{a.s.}$$

Let $\eta(t)$ be an RCLL version of $E(\eta|\mathcal{F}_t^w)$. Then the martingale $\{\eta(t),\mathcal{F}_t^w\,;\,t\varepsilon\underline{T}\}$ has a continuous modification.

An immediate consequence of this is:

Proposition 5.1.1: The information structure $\mathbb{F}^w$ generated by a standard Brownian motion is continuous.

Proof: By Lemma 5.1.1 we know that any martingale adapted to $\mathbb{F}^w$ has a continuous modification. It then follows from Proposition 4.1 that $\mathbb{F}^w$ is continuous.
Q.E.D.

Remark: The filtration generated by a multidimensional Brownian motion is also continuous.

### 5.2 The Autarchy Example

Let us keep all the setup from the previous sections except to assuming the following. Let there be one representative agent whose preferences can be represented with a time-additive "state dependent expected utility function" defined on the net trade space V.

Formally, the representative agent at time zero maximizes the expected utility:

$$U^*(r,x) = f(r) + \int_\Omega g^*(x(\omega),\omega)P(d\omega) \quad ,$$

where $f: R \to R$ is strictly increasing and concave, and where $g^*: R \times \Omega \to R$ is such that

1. $g^*(y,\omega)$ is strictly increasing and concave in y P-almost surely;
2. for each y, $g^*(y,\omega)$ is integrable on $(\Omega,\mathcal{F},P)$; and
3. if we denote the left-hand derivative of $g^*$ with respect to its first argument by $D^-g^*$, then $D^-g^*(0,\omega)$ is integrable on $(\Omega,\mathcal{F},P)$.[7]

It should be noted here that the representative agent's preferences may not be τ-continuous. Of course τ-continuous preferences are sufficient for the results in previous sections but are certainly not necessary, as will be shown. Now we define U to be a modification of $U^*$ as follows:

$$U(r,x) = f(r) + \int_\Omega g(x(\omega),\omega)P(d\omega) \quad ,$$

where

$$\begin{aligned} g(y,\omega) &= g^*(y,\omega) && \text{for } y > 0 \\ &= g^*(0,\omega) + D^-g^*(0,\omega)y && \text{for } y \leq 0 \, . \end{aligned}$$

Proposition 5.2.1: $U: R \times X \to R$ is τ-continuous.

Proof: See Appendix I.

Q.E.D.

Proposition 5.2.2: If we define $\delta': R_+ \times X_+ \to R$ as

$$\delta'(r,x) = \lim_{\alpha \to 0+} \frac{U(\alpha r, \alpha x) - U(0,0)}{\alpha} \quad ,$$

then δ' is well-defined and is a τ-continuous and K-strictly positive linear functional on $R_+ \times X_+$ and has the following form:

$$\delta'(r,x) = D^+f(0)r + \int_\Omega D^+g(0,\omega)x(\omega)P(d\omega)$$

$$\equiv D^+f(0)r + \psi'(x) \ , \tag{5.2.1}$$

where K denotes the cone with the origin deleted from $R_+ \times X_+$, $D^+g$ denotes the right-hand-derivative of g with respect to its first argument, $D^+f$ denotes the right-hand-derivative of f, and $\psi'$ is a linear functional on $X_+$.

Proof: See Appendix I.

Q.E.D.

Note that $d_0 \varepsilon X_{++}$, so $\psi'(d_0)$ is well-defined and is strictly positive. It can easily be seen from (5.2.1) that $\delta'$ has an extension to all of $R \times X$ that is $\tau$-continuous, K-strictly positive, and has the same form as $\delta'$, $\delta$: $R \times X \to R$:

$$\delta(r,x) = D^+f(0)r + \int_\Omega D^+g(0,\omega)x(\omega)P(d\omega)$$

$$\equiv D^+f(0)r + \psi(x) \quad .$$

Choosing the $0^{th}$ claim as the numeraire and defining $\delta^*: R \times X \to R$ by

$$\delta^*(r,x) = \frac{D^+f(0)}{\psi(d_0)}\, r + \frac{\psi(x)}{\psi(d_0)} \quad .$$

As defined, $\delta^*$ is a $\tau$-continuous, K-strictly positive linear functional on $R \times X$. Now we are ready to prove the main result in this section.

Theorem 5.2.1: $(0,0) \varepsilon R \times X$ solves the following program:

$$\max_{(r,x)\varepsilon R\times X} \quad U^*(r,x)$$

$$\text{s.t.} \qquad \delta^*(r,x) \leqslant 0$$

Proof: We first claim that $(0,0) \varepsilon R \times X$ solves

$$\max_{(r,x)\varepsilon R\times X} \quad U(r,x)$$

$$\text{s.t.} \qquad \delta^*(r,x) \leqslant 0 \ .$$

The fact that (0,0) is feasible should be obvious, since $\delta^*(0,0) = 0$. Now select any $(r,x) \varepsilon R \times X$ such that $\delta^*(r,x) \leq 0$. We consider

$$U(r,x) - U(0,0) = f(r) - f(0) + \int_\Omega (g(x(\omega)),\omega) - g(0,\omega))P(d\omega).$$

By concavity we have:

$$f(r) - f(0) \leq D^+f(0)r \quad ,$$

and

$$g(x(\omega),\omega) - g(0,\omega) \leq D^+g(0,\omega)x(\omega) \quad \text{P-a.s.}$$

These imply that

$$\begin{aligned} U(r,x) - U(0,0) &\leq D^+f(0)\, r + \int_\Omega D^+g(0,\omega)x(\omega)P(d\omega) \\ &= \psi(d_0)\,\delta^*(r,x) \leq 0 \; . \end{aligned} \tag{5.2.2}$$

Thus $U(0,0) \geq U(r,x)\ \forall\ (r,x)\varepsilon R \times X$ such that $\delta^*(r,x) \leq 0$.

Next, again let $(r,x)\varepsilon R \times X$ such that $\delta^*(r,x) \leq 0$. Then we have

$$\begin{aligned} U^*(r,x) - U^*(0,0) &= f(r) - f(0) + \int_\Omega (g^*(x(\omega),\omega) - g^*(0,\omega))P(d\omega) \\ &\leq f(r) - f(0) + \int_\Omega (g(x(\omega),\omega) - g(0,\omega))P(d\omega) \\ &= U(r,x) - U(0,0) \\ &\leq 0 \; . \end{aligned}$$

The second line follows from the definition of g and the last inequality, which was to be shown, follows from (5.2.2).

Q.E.D.

From this theorem we know that if all claims are marketed and if their prices are given by $\delta^*$, then the representative agent's preferences on net trades are maximized at (0,0). This theorem also implies that even when the space of marketed claims is not equal to X, but their current prices are given by $\delta^*$, $(0,0)\varepsilon R \times X$ is still a maximum. Now what is left to do is to define a price system S and to show that any claim generated by an admissible trading strategy $\theta\varepsilon\Theta$ has a current price given by $\delta^*$.

Let us put $\xi(\omega) = D^+g(0,\omega)d_0(\omega)/\psi(d_0)$ and $\forall A \varepsilon \mathcal{F}$ let $Q(A) = \int_A \xi(\omega)P(d\omega)$. Then Q is a probability measure on $(\Omega,\mathcal{F})$ equivalent to P. It is a probability measure since $Q(\Omega) = \int_\Omega (D^+g(0,\omega)d_0(\omega)/\psi(d_0))P(d\omega) = \psi(d_0)/\psi(d_0) = 1$, and it is equivalent to P since $\xi$ is strictly positive. It then follows that $\psi(x)/\psi(d_0) = E^*(x/d_0)$ where $E^*(\cdot)$ is the expectation operator under Q. Define

$$S_n(t) = E^*[\, d_n^* | \mathcal{F}_t \,] , \quad \forall t \varepsilon \underline{T} , \text{ and } n=0,.,\ldots,N ,$$

where $d_n^* = d_n/d_0$. It is straightforward to verify that the price system S is consistent with $\delta^*$, since by construction $S_n(0) = E^*(d_n^*) = E^*(d_n/d_0) = \psi(d_n)/\psi(d_0)$. Furthermore, $E^*[\,\theta(T)\cdot S(T)\,] = \theta(0)\cdot S(0)$ for any $\theta\varepsilon\Theta$, since S is a vector bounded martingale under Q by construction and $\theta$ is also bounded. (For this point, see Lemma 3.2.) Thus the current prices for marketed claims are also consistent with $\delta^*$. S is therefore an equilibrium price system with $\theta_n(t) = 0$, $\forall t\varepsilon\underline{T}$ and $n=0,1,\ldots,N$, being the equilibrium trading strategy.

Up to now, the information structure generated by a standard Brownian motion with continuous sample paths has not come into the story. We have just proved that there exists an equilibrium in our economy with a representative agent, and the equilibrium price system, with the $0^{th}$ claim chosen to be the numeraire, is a vector bounded martingale under Q.

If the information structure is the Brownian filtration, $\mathbb{F}^w$, we can show a bit more. First, it follows from Proposition 5.1.1 that S has a continuous modification, since $\mathbb{F}^w$ is a continuous information structure. Next, we want to show that S can be represented as a vector Ito integral. Letting $\xi(t)$ be an RCLL version of $E[\xi|\mathcal{F}_t^w]$, $\{\xi(t), \mathcal{F}_t^w; t\varepsilon\underline{T}\}$ is a martingale on $(\Omega,\mathcal{F},P)$. By Lemma 5.1.1, there exists a $\phi\varepsilon\Phi$ such that

$$\begin{aligned}\xi(t) &= E(\xi) + \int_0^t \phi(s)\,dw(s) \quad \text{a.s.}\\ &= 1 + \int_0^t \phi(s)\,dw(s) \quad \text{a.s.}\end{aligned} \tag{5.2.3.}$$

for $t\varepsilon\underline{T}$. Since $\xi$ is strictly positive so is $\xi(t)$ for all $t\varepsilon\underline{T}$. Recall that $\{S(t),\mathcal{F}_t^w; t\varepsilon\underline{T}\}$ is a martingale on $(\Omega,\mathcal{F},Q)$. Then by Lemma 4.1, $\{S(t)\xi(t),\mathcal{F}_t^w; t\varepsilon\underline{T}\}$ is a martingale on $(\Omega,\mathcal{F},P)$. Consider, say, claim n. Again by Lemma 5.1, we know there exists a $\hat{\phi}\varepsilon\Phi$ such that

$$\begin{aligned}S_n(t)\xi(t) &= E(S_n(T)\xi) + \int_0^t \hat{\phi}(s)\,dw(s) \quad P-\text{a.s.}\\ &= S_n(0) + \int_0^t \hat{\phi}(s)\,dw(s) \quad P-\text{a.s.}\end{aligned} \tag{5.2.4}$$

Eqs. (5.2.3) and (5.2.4) imply that the equilibrium price system S can be represented as a vector Ito integral. This follows directly from the Ito's Lemma and the fact that $\xi(t)$ is strictly positive for all $t\varepsilon\underline{T}$ and can be taken to be continuous.

Remark: Note that in showing the Ito-integral-representation property of S, the fact that there is only one representative agent in the economy is nowhere utilized. It should therefore be clear that that property will be valid in an economy with heterogeneous agents when an equilibrium indeed exists.

## 8. Other Information Structures

In Section 4, a definition for the continuity of an information structure was given. Such an information structure was then shown to imply that the relative prices of an equilibrium price system must be continuous with probability one. Two other information structures will be considered in this section. By considering these, we hope not only to associate different price behaviors with different information structures, but also to shed some light on the following question: what does continuity of an information structure buy us? It will ultimately be shown that a continuous information structure is one on which "no events can take us by surprise," in a sense to be explained. A large part of the discussion in this section can be found in the literature of probability theory. (See Dellacherie and Meyer [12], for example.) Efforts are made, however, to recast definitions and results in the context of the present economic model.

Recall that an information structure is a right-continuous filtration $\mathbb{F}$. One may wonder whether a filtration with $\mathcal{F}_t = \mathcal{F}_{t-} \equiv \bigvee_{s<t} \mathcal{F}_s$ for $t \in [0,T]$ is a continuous information structure.

Definition 6.1: An information structure $\mathbb{F}$ on $(\Omega,\mathcal{F},P)$ is said to be semi-continuous if $\mathcal{F}_t = \mathcal{F}_{t-}$ $\forall\, t \in \underline{T}$. (We have used the convection that $\mathcal{F}_0 = \mathcal{F}_{0-}$.)

If the information structure is semi-continuous then, for every deterministic $t \in \underline{T}$, agents do not know more than they did an instant before, except possibly on a P-null set. (Here we should recall that $\mathcal{F}_0$ is assumed to contain all the P-negligible sets.) Let $P^*$ be a probability measure on $(\Omega,\mathcal{F})$, which is equivalent to P. Then it is clear that $\mathbb{F}$ is also semi-continuous on $(\Omega,\mathcal{F},P^*)$, since P and $P^*$ have the same null sets. With this kind of information structure, a similar but significantly weaker form of Proposition 4.2 is available.

Proposition 6.1: Let $Z = \{Z(t), \mathcal{F}_t ; t\in\underline{T}\}$ be an adapted martingale on $(\Omega,\mathcal{F},P)$. If $\mathbb{F}$ is semi-continuous, then $Z(t) = Z(t-)$ P-almost surely for every $t\in\underline{T}$.

Proof: Theorem 4 in Chapter VI of Meyer [34] states that if Z is a martingale, then

$$Z(t-) = E[Z(t)|\mathcal{F}_{t-}] \text{ a.s.}$$

By the definition of a semi-continuous information structure we have $\mathcal{F}_t = \mathcal{F}_{t-}$, therefore

$$Z(t-) = E[Z(t)|\mathcal{F}_t] = Z(t) \quad \text{a.s.},$$

where the convention that $Z(0-) = Z(0)$ is used.

Q.E.D.

Now it is straightforward to prove the following:

Theorem 6.1: Let t belong to $[0,T]$. If the agents' information structure is semi-continuous, the equilibrium price system is continuous at t, P-almost surely.

Proof: S is a martingale on $(\Omega,\mathcal{F},Q)$, hence $S(t-) = S(t)$ Q-a.s. This is true since $\mathbb{F}$ is a semi-continuous information structure on $(\Omega,\mathcal{F},P)$ if and only if it is one on $(\Omega,\mathcal{F},Q)$, and since P and Q are equivalent. For the same reason we have $S(t-) = S(t)$ P-a.s. Q.E.D.

Theorem 6.1 says that the equilibrium price system is continuous at any $t\in\underline{T}$ P-almost surely, but it does not say whether S is a continuous process with probability one; that is, whether there exists a P-null set N such that $t \to S(\omega,t)$ is a continuous mapping for all but the set N in $(\Omega,\mathcal{F})$. An uncountable union of null sets is not necessarily a null set itself, and may not even be measurable. In the following example, it will be shown that when $\mathbb{F}$ is semi-continuous, the equilibrium price system can jump with strictly positive probability.

Let us consider the economy in Section 5, but with an information structure now generated by a standard Poisson process with parameter 1 starting from 0 (rather than a standard Brownian motion). Denote this Poisson process by

$\hat{N} = \{\hat{N}(t), \mathcal{F}_t ; t\varepsilon T\}$. The filtration $\mathbb{F}$ is semi-continuous, since at each time $t\varepsilon\underline{T}$ the probability of a jump is zero. Putting $Y(t) = \hat{N}(t) - t$, $Y = \{Y(t), \mathcal{F}_t ; t\varepsilon\underline{T}\}$ is a martingale. It is continuous at t almost surely for every $t\varepsilon\underline{T}$ because $\hat{N}$ is. Now consider the optional random variable

$$J(\omega) = \inf \{t\varepsilon\underline{T} : \hat{N}(\omega,t) = 1\} \quad \text{on } B$$

$$= \infty \quad \text{on } \Omega\setminus B ,$$

where $B = \{\omega\varepsilon\Omega : \hat{N}(\omega,t) \neq 0\}$. The probability of event B is $1 - e^{-T}$, which is strictly positive. The martingale Y will experience a jump at J when $J \neq \infty$. Or equivalently, the sample paths of Y are discontinuous on a set of strictly positive probability. And therefore Y is not a continuous process with probability one. A characterization for martingales adapted to the filtration generated by a Poisson process, due to Davis [9], is available.

<u>Lemma 6.1</u>: Let $\mathbb{F}$ be the filtration generated by $\hat{N}$. Suppose $Z = \{Z(t), \mathcal{F}_t ; t\varepsilon\underline{T}\}$ is a martingale on $(\Omega,\mathcal{F},P)$. Then there exists a predictable process $\eta(t)$ with

$$E [ \int_0^T |\eta(s)| ds ] < \infty$$

such that

$$Z(t) = Z(0) + \int_0^t \eta(s) dY(s) \quad \text{a.s.}$$

where we recall that $Y(t) = \hat{N}(t) - t$.

From this lemma, it is evident that any martingale adapted to the Poisson filtration can only have jumps when the Poisson process jumps. But a martingale does not have to jump when the Poisson process jumps, since on $\{J < \infty\}$, $\eta(J)$ may be zero. A good example for this is a constant process $Z(t) = Z(0)$ for all $t\varepsilon\underline{T}$. It is a martingale and is a continuous process.

Recall from Section 5 that the price for claim n at time t is $E^*[d_n^*|\mathcal{F}_t]$. Equivalently, $S_n(t) = E [ d_n^*\xi|\mathcal{F}_t ] /\xi(t)$. By Lemma 6.1 we know that there exist

predictable processes $\eta_1$ and $\eta_2$ with $E[\int_0^T |\eta_1(s)|ds] < \infty$ and $E[\int_0^T |\eta_2(s)|ds] < \infty$ such that

$$\xi(t) = 1 + \int_0^t \eta_1(s)dY(s) \text{ a.s. } \forall t\varepsilon \underline{T}$$

and

$$S_n(t)\xi(t) = S_n(0) + \int_0^t \eta_2(s)dY(s) \text{ a.s. } \forall t\varepsilon \underline{T} .$$

Therefore we have:

$$S_n(t) = \frac{S_n(0) + \int_0^t \eta_2(s)dY(s)}{1 + \int_0^t \eta_1(s)dY(s)}$$

$$= \frac{S_n(0) + \int_0^t \eta_2(s)d\hat{N}(s) - \int_0^t \eta_2(s)ds}{1 + \int_0^t \eta_1(s)d\hat{N}(s) - \int_0^t \eta_1(s)ds} \text{ a.s.}$$

From the above equation we know $S_n$ can jump only when $\hat{N}$ jumps. Between jumps, $S_n$ is absolutely continuous. In the interval $[0,T]$, $\hat{N}$ jumps on a set of strictly positively probability. Thus it can happen that $S_n$ jumps on a set of strictly positive probability and is not a continuous process. $S_n$ can still be continuous, however, when $\hat{N}$ jumps. An extreme example for this phenomenon is that the payoff structure of claim n is proportional almost surely to that of the $0^{th}$ claim.

Before continuing, some remarks are in order. In the previous discussion, each process was a real-valued function defined on $\Omega \times [0,T]$. In the sequel, for ease of discussion, we shall use the following convention. A process Z is defined on $\Omega \times ([0,\infty] \cup \{0-\})$, with $Z(t) = Z(T)$ for $t \geq T$ and $Z(0-) = Z(0)$. The information structure is then $\mathbb{F} = \{\mathcal{F}_t ; t\varepsilon[0,\infty] \cup \{0-\}\}$, with $\mathcal{F}_{0-} = \mathcal{F}_0$ and $\mathcal{F}_t = \mathcal{F}_T$ for $t \geq T$.

A stopping time J is a positive random variable defined on $(\Omega,\mathcal{F},P)$ which can take on the value $\infty$ with a strictly positive probability and for which $\{J \leq t\}\varepsilon\mathcal{F}_t$ for every t. A stopping time J is said to be predictable if there exists an increasing sequence of stopping times $(J_n)$ with $J_n \leq J$ P-a.s. such that on the set $\{\omega\varepsilon\Omega : J(\omega) > 0\}$ we have $J_n < J$ and $J_n \to J$ almost everywhere except possibly

on a P-measure zero set. This sequence of stopping times is said to <u>foretell</u> J. If we interpret J to be the first time an event B happens, this event B is foretellable by a sequence of events except possibly on a set of probability zero. (Here we note that the event $\{J = 0\}$ is either of probability zero or one by the assumption that $\mathcal{F}_0$ is almost trivial.) As put by Dellacherie and Meyer [12]: "we are forewarned by a succession of precursory signs, of the exact time the phenomenon will occur."

<u>Definition 6.2</u>: An information structure $\mathbb{F}$ is said to be <u>quasi-continuous</u> if for any predictable stopping time J and any increasing sequence of stopping times $(J_n)$ foretelling it, we have:

$$\mathcal{F}_J = \bigvee_{n=0}^{\infty} \mathcal{F}_{J_n} .$$

If $\mathbb{F}$ is quasi-continuous, it is then semi-continuous since and deterministic t is a predictable stopping time. We note further that the definition of predictable stopping times depends only on the null sets of the underlying probability space. It should, therefore, be evident that the definition of a quasi-continuous information structure is also invariant under a substitution of an equivalent probability measure. Under the above definition, we have a characterization of adapted martingales which is stronger than Proposition 6.1, but is still weaker than Proposition 3.2.

<u>Proposition 6.2</u>: Let Z be an adapted martingale on $(\Omega,\mathcal{F},P)$ and J a predictable stopping time. If $\mathbb{F}$ is quasi-continuous, then $Z(J-) = Z(J)$ P-almost surely.

<u>Proof</u>: Let $(J_n)$ be an increasing sequence of stopping times that foretells J. We have by VI.T14 of Meyer [34]:

$$Z(J) = E[\, Z(T) \,|\, \mathcal{F}_J \,] = E[\, Z(T) \,|\, \bigvee_{n=0}^{\infty} \mathcal{F}_{J_n} \,]$$

$$= \lim_{n\to\infty} E[Z(T)|\mathcal{F}_{J_n}] = \lim_{n\to\infty} Z(J_n) = Z(J-) \text{ P-a.s.}$$

Q.E.D.

The left-continuity of a martingale at predictable stopping times is equivalent to continuity at them, since we always take a martingale to be an RCLL process.

Theorem 6.2: If F is quasi-continuous, then the equilibrium price system S is continuous at a predictable stopping time with probability one.

Proof: S is a vector martingale on $(\Omega,\mathcal{F},Q)$, so

$$S(J) = S(J-) \quad Q\text{-a.s.}$$

for any predictable stopping time J on $(\Omega,\mathcal{F},Q)$. By the equivalence between P and Q, we get

$$S(J-) = S(J) \quad P\text{-a.s.}$$

But a stopping time is predictable on $(\Omega,\mathcal{F},P)$ if and only if it is on $(\Omega,\mathcal{F},Q)$. Hence, J is predictable on $(\Omega,\mathcal{F},P)$. Thus, by Eq. (5.1), the result follows.

Q.E.D.

One consequence of Theorem 6.2 is that an equilibrium price system can only jump at non-predictable stopping times when the information structure is quasi-continuous. Putting it differently, an equilibrium price system can not jump at events that are fully anticipated. It has been shown by Meyer [32] that any information structure generated by a process that is continuous at predictable stopping times and has the strong Markov property is quasi-continuous. The Poisson filtration discussed earlier in this section fits into this category and is thus not only semi-continuous but indeed quasi-continuous.

Remark: From the results in Meyer [32], it can be easily seen that a filtration generated by a continuous process having the strong Markov property is a continuous information structure. Therefore an information structure generated by a diffusion process is continuous.

After a substantial detour, the next theorem will achieve the objective proclaimed at the very beginning of this section. Several definitions are in order.

Definition 6.3: A stopping time J is said to be totally inaccessible if it is not P-almost surely infinity and if

$$P(J = J' < \infty) = 0$$

for all predictable stopping times J'.

Definition 6.4: A stopping time J is said to be inaccessible if there exists a totally inaccessible stopping time J' such that

$$P(J = J' < \infty) > 0 .$$

Definition 6.5: A stopping time J is said to be accessible if it is not inaccessible.

From the definitions above, it is clear that a predictable stopping time is certainly accessible, but an accessible stopping time is not necessarily predictable. The following lemma is, however, available :

Lemma 6.2: The set of predictable stopping times coincides with the set of accessible times if and only if the information structure is quasi-continuous.

Proof: See Section 83 of Chapter IV, Dellacherie and Meyer [12].

Q.E.D.

Theorem 6.3: $\mathbb{F}$ is continuous if and only if all stopping times are predictable.

Proof: First suppose that all stopping times are predictable. It then follows that the set of predictable stopping times and the set of accessible stopping times are identical. Therefore, the information structure is quasi-continuous by Lemma 6.2. Armed with Proposition 4.2, it suffices to show that any martingale on $(\Omega,\mathcal{F},P)$ adapted to F has a continuous modification. Let Z be such a martingale. By definition Z is RCLL. Define a predictable process Y by:

$$Y(t) = Z(t-) .$$

Y is then LCRL. Since all stopping times are predictable, it therefore follows

from Proposition 6.2 that

$$Z(J) = Z(J-) \quad P\text{-a.s.}$$

for any stopping time J. By construction, this also implies that

$$Z(J) = Y(J) \quad P\text{-a.s.}$$

Result IV.86 in Dellacherie and Meyer [12] says that if two optional processes are equal almost surely at every stopping time, then they are indistinguishable. Note that predictable processes and RCLL processes are both optional. (See IV.60 and IV.67 in Dellacherie and Meyer [12].) Hence, Y and Z are indistinguishable processes. Recall that Z is RCLL and Y is LCRL. It then follows that Z must be indistinguishable from a continuous process, which was to be shown.

Conversely, suppose $\mathbb{F}$ is a continuous information structure and there exists a stopping time J that is not predictable. Then it is possible to construct a totally inaccessible stopping time J' from J, and a martingale Z, which has a jump of size $\underline{1}$ at J'. The former follows from IV.81 of Dellacherie and Meyer [12] and the latter follows from VII.T46 of Meyer [34]. This contradicts Proposition 4.2, and all stopping times must therefore be predictable.

Q.E.D.

A continuous information structure is one on which all stopping times are predictable. Equivalently, an information structure on which no event takes us by surprise is continuous. The above fact was also discovered independently by J. Michael Harrison and Richard Pitbladdo (in preparation).

## 7. Concluding Remarks

This paper has addressed the following questions. First, what conditions on agents' preferences and information structure will ensure that equilibrium prices have continuous sample paths, if an equilibrium exists? In particular, when can equilibrium prices be represented as Ito integrals? Second, can the existence of an equilibrium be established? Third, can the behavior of equilibrium prices be characterized under different information structures? Quite satisfactory answers are found for the first question. In response to the second question, the existence of an equilibrium is established in the case of an autarchy. Some expository observations are made concerning the third question: given that agents' preferences are "continuous" enough, the manner in which information arrives solely determines the behavior of asset prices.

If consumption occurs continuously, when will the cumulative consumption be absolutely continuous, guaranteeing that a consumption rate actually exists? Moreover, does a continuous information structure generated by a Brownian motion imply that the consumption rate, if it indeed exists, is an Ito integral? These two questions are foundational issues related to the consumption CAPM of Breeden [1979]. Throughout this paper agents consume at two time points, 0 and T, without intermediate consumption, so these issues cannot be addressed here. For an exploratory study in that direction see Huang [22].

# APPENDIX I: Proofs of Propositions 5.2.1 and 5.2.2

Proposition 5.2.1: $U: R \times X \to R$ is $\tau$-continuous.

Proof: $U: R \times X \to R$ has the form:

$$U(r,x) = f(r) + \int_{\Omega} g(x(\omega),\omega)P(d\omega) \quad .$$

The product topology $\tau$ is a topology of pointwise convergence. So it suffices to show that f is continuous in the Euclidean norm topology on R and $\int_{\Omega} g(x(\omega),\omega)P(d\omega)$ is continuous in the $L^1$-Mackey topology on X. The fact that f is continuous in the Euclidean norm topology is obvious, since f is a concave function on R and therefore continuous on R. In fact, it is absolutely continuous over any closed interval.

Next let $(x^{\alpha})$ be a net in X with $(x^{\alpha}) \to x$ in the $L^1$-Mackey topology. Define

$$\bar{x}^{\alpha}(\omega) = x(\omega) + |x^{\alpha}(\omega) - x(\omega)| \quad ,$$

and

$$\underline{x}^{\alpha}(\omega) = x(\omega) - |x^{\alpha}(\omega) - x(\omega)| \quad .$$

Note first that if $x^{\alpha} \to x$ in the $L^1$-Mackey topology, then $|x^{\alpha} - x| \to 0$ in the same topology.[8] Thus $\bar{x}^{\alpha} \to x$ and $\underline{x}^{\alpha} \to x$ in the $L^1$-Mackey topology. Secondly, by monotonicity we have:

$$\begin{aligned} &| \; g(x(\omega),\omega) - g(x^{\alpha}(\omega),\omega) \; | \\ &\leq g(\bar{x}^{\alpha}(\omega),\omega) - g(\underline{x}^{\alpha}(\omega),\omega) \\ &= g(\bar{x}^{\alpha}(\omega),\omega) - g(x(\omega),\omega) + g(x(\omega),\omega) - g(\underline{x}^{\alpha}(\omega),\omega) \quad . \end{aligned}$$

Hence,

$$\begin{aligned} &| \int_{\Omega} (g(x(\omega),\omega) - g(x^{\alpha}(\omega),\omega))P(d\omega) \; | \\ &\leq \int_{\Omega} (g(\bar{x}^{\alpha}(\omega),\omega) - g(x(\omega),\omega))P(d\omega) + \int_{\Omega} (g(x(\omega),\omega) - g(\underline{x}^{\alpha}(w),\omega))P(d\omega) \quad . \end{aligned}$$

Therefore it is sufficient to show that for nets $(x^\alpha)$ with $x^\alpha \leq x$, $x^\alpha \to x$ and for nets $(x^\alpha)$ with $x^\alpha \geq x$, $x^\alpha \to x$, that

$$\int_\Omega g(x^\alpha(\omega),\omega)P(d\omega) \to \int_\Omega g(x(\omega),\omega)P(d\omega) \quad .$$

Now let $(x^\alpha)$ be a net in X, with $x^\alpha \geq x$, and $x^\alpha \to x$. By concavity we have:

$$\int_\Omega (g(x^\alpha(\omega),\omega) - g(x(\omega),\omega))P(d\omega) \leq \int_\Omega D^- g(0,\omega)(x^\alpha(\omega) - x(\omega))P(d\omega) \quad .$$

Since $x^\alpha \to x$ in the $L^1$-Mackey topology, $x^\alpha$ also converges to x in the weak topology on X generated by $L^1(\Omega,\mathcal{F},P)$. That is, for any $y \in L^1(\Omega,\mathcal{F},P)$,

$$\int_\Omega x^\alpha(\omega)y(\omega)P(d\omega) \to \int_\Omega x(\omega)y(\omega)P(d\omega) \quad .$$

Recalling the assumption that $D^- g(0,\omega)$ belongs to $L^1(\Omega,\mathcal{F},P)$, we get:

$$\int_\Omega D^- g(0,\omega)(x^\alpha(\omega) - x(\omega))P(d\omega) \to 0 \quad ,$$

which was to be shown. The case where $x^\alpha \leq x$ and $x^\alpha \to x$ is identical, so we omit the details for that case. Q.E.D.

<u>Proposition 5.2.2</u>: $\delta'$ is a $\tau$-continuous and K-strictly positive linear functional on $R_+ \times X_+$ and has the form:

$$\delta'(r,x) = D^+ f(0)r + \int_\Omega D^+ g(0,\omega)x(\omega)P(d\omega) \quad .$$

<u>Proof</u>: Let $(r,x) \in R_+ \times X_+$. By definition of $\delta'$:

$$\delta'(r,x) = \lim_{\alpha \to 0+} \frac{U(\alpha r , \alpha x) - U(0,0)}{\alpha}$$

$$= \lim_{\alpha \to 0+} \frac{f(\alpha r) - f(0) + \int_\Omega (g(\alpha x(\omega),\omega) - g(0,\omega))P(d\omega)}{\alpha}$$

if the limit exists. We first claim that

$$\lim_{\alpha \to 0+} \frac{f(\alpha r) - f(0)}{\alpha} = D^{+}f(0)r \quad , \tag{A.1}$$

where $D^{+}f(0)$ denotes the right-hand-derivative of f at zero. To verify this, recall from calculus that if f is concave,

$$f(r) = f(0) + D^{+}f(0)r + o(|r|) \quad ,$$

where $\lim_{|r| \to 0} o(|r|)/ r = 0$. (A.1) follows directly from (A.2). Next, by concavity, we have:

$$\frac{g(\alpha x(\omega),\omega) - g(0,\omega)}{\alpha} \leqslant D^{-}g(0,\omega)x(\omega) \quad , \quad \text{a.s.}$$

By assumption (3) in Section 4.2 we know that $D^{-}g(0,\omega)$ is integrable and that $x(\omega)$ is essentially bounded, so $D^{-}g(0,\omega)x(\omega)$ is integrable. Then it follows from the Lebesgue convergence theorem that

$$\lim_{\alpha \to 0+} \int_{\Omega} \frac{g(\alpha x(\omega),\omega) - g(0,\omega)}{\alpha} P(d\omega)$$

$$= \int_{\Omega} D^{+}g(0,\omega)x(\omega)P(d\omega) < \infty \; .$$

Hence $\delta'(r,x)$ is well-defined and is equal to

$$D^{+}f(0)r + \int_{\Omega} D^{+}g(0,\omega)x(\omega)P(d\omega) \quad .$$

Now we are left to show that $\delta'$ is a $\tau$-continuous and K-strictly positive linear functional on $R_{+} \times X_{+}$. That $\delta'$ is linear and K-strictly positive follows from the linearity of the integral and the strict monotonicity of the agent's preferences. If $D^{+}g(0,\omega)$ can be shown to be integrable, then we are done, since $\int_{\Omega} D^{+}g(0,\omega)x(\omega)P(d\omega)$ will then be a continuous linear function on $R_{+} \times X_{+}$. Note

that again by concavity and monotonicity we have:

$$0 \leqslant D^{+}g(0,\omega) \leqslant D^{-}g(0,\omega) \quad \text{a.s.},$$

and the integrability of $D^{+}g(0,\omega)$ follows from the integrability of $D^{-}g(0,\omega)$. Therefore $\delta'$ is a $\tau$-continuous and K-strictly positive linear functional on $R_{+} \times X_{+}$ with the desired form. Q.E.D.

FOOTNOTES

[1] Here we should mention the work of Cox, Ingersoll, and Ross [8]. They modeled production explicitly in an autarchy economy. Once it is assumed that there exists a smooth solution to the representative agent's control problem, the equilibrium asset price processes are Ito integrals.

[2] A Borel field is said to be almost trivial if it contains only sets of P-measure one or zero.

[3] This is the product Borel-field generated by $\mathscr{F}$ and the Euclidean Borel-field on [0,T].

[4] For a brief discussion concerning this topology, see Bewley [2].

[5] For a discussion of conditional expectations with respect to the Borel fields, see Chapter 9 of Chung [6].

[6] For this point, see Harrison and Pliska [21].

[7] Here we should note that the left-hand derivatives and the right-hand derivatives of a concave function exist everywhere, and that preferences on net trades will in general be state dependent as long as agents' endowments are random.

[8] See the proof of the theorem in Appendix II of Bewley [2].

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